Static plane symmetric relativistic fluids and empty repelling singular boundaries

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Abstract. We present a detailed analysis of the general exact solution of Einstein's equation corresponding to a static and plane symmetric distribution of matter with density proportional to pressure. We study the geodesics in it and we show that this simple spacetime exhibits very curious properties. In particular, it has a free of matter repelling singular boundary and all geodesics bounce off it.

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1. Introduction

Because of the complexity of Einstein's field equations, one cannot find exact solutions except in spaces of rather high symmetry—very often with no direct physical application. Nevertheless, exact solutions can give an idea of the qualitative features that could arise in General Relativity and so, of possible properties of realistic solutions of the field equations.

In this paper we want to illustrate some curious features of gravitation by means of a simple solution: the gravitational field of a static plane symmetric relativistic perfect fluid.

We have recently pointed out that solutions of Einstein's equation presenting an empty (free of matter) repelling boundary where spacetime curvature diverges occur [1], and we called this kind of singularities *white walls*. These singularities are not the sources of the fields but they arise owing to the attraction of distant matter.

The solution we described in [1] is the gravitational field of a static homogenous distribution of matter with plane symmetry lying below $z = 0$. Because of the symmetry required, the exterior gravitational field turns out to be Taub's plane vacuum solution [2].

Although the properties of Taub's plane solution have been known for several years and perhaps due to the belief that singularities are always the sources of vacuum solutions, this solution has been usually associated to matter with negative mass (see [3, 4] and references therein). For example, the authors of Ref. [4] suggest that it must be interpreted as representing the exterior gravitational field of a planar shell

with *infinite negative* mass density at the singularity, whereas we argue that Taub's plane solution is also the external gravitational field of ordinary (i.e., with nonnegative density and pressure) matter sitting at negative values of z and that the singularity that arises high above is due to the attraction of the distant matter [1].

Thus, it would be worth finding an exact solution of Einstein's equation corresponding to a distribution of ordinary matter presenting a singularity of this kind.

The aim of this paper is to show that the solution found by Collins [5] for a static and plane symmetric relativistic perfect fluid obeying an equation of state such that ρ and p are proportional to each other (see also [6] and for the case $\rho = p$ [7]) exhibits such a property.

In Sec. II we present a simple and direct derivation of Collins's solution, as well as a detailed analysis of it. In Sec. III we study the geodesics in it.

Throughout this paper, we adopt the convention in which the spacetime metric has signature $(- \ + \ + +)$, the system of units in which the speed of light $c = 1$ and Newton's gravitational constant $G = 1$.

2. The Collins solution

We want to find the solution of Einstein's equation corresponding to a static and plane symmetric distribution of matter with plane symmetry. That is, it must be invariant under translations in the plane and under rotations around its normal. The matter we will consider is a perfect fluid satisfying the equation of state ‡

$$\rho = \eta\, p\,, \tag{1}$$

where η is an arbitrary constant—for ordinary matter $0 < p < \rho$ and so $\eta > 1$. The stress-energy tensor is

$$T_{ab} = (\rho + p)\, u_a u_b + p\, g_{ab}\,, \tag{2}$$

where u^a is the velocity of fluid elements.

Due to the plane symmetry and staticity, following [2] we can find coordinates (t, x, y, z) such that

$$ds^2 = -e^{2U(z)}\left(dt^2 - dz^2\right) + e^{2V(z)}\left(dx^2 + dy^2\right)\,. \tag{3}$$

That is, it is the more general metric admitting the Killing vectors ∂_x, ∂_y, $x\partial_y - y\partial_x$ and ∂_t.

The non-identically vanishing components of the Einstein tensor are

$$G_{tt} = 2\,\partial_z U\,\partial_z V - 3\,(\partial_z V)^2 - 2\,\partial_z\partial_z V, \tag{4}$$

$$G_{xx} = G_{yy} = e^{-2U+2V}\left((\partial_z V)^2 + \partial_z\partial_z V + \partial_z\partial_z U\right)\,, \tag{5}$$

$$G_{zz} = \partial_z V\,(2\,\partial_z U + \partial_z V\,)\,. \tag{6}$$

On the other hand, since the fluid must be static, $u^a = (e^{-U}, 0, 0, 0)$, so

$$T_{ab} = \mathrm{diag}\left(\rho\, e^{2U}, p\, e^{2V}, p\, e^{2V}, p\, e^{2U}\right)\,, \tag{7}$$

where ρ and p can depend only on the z-coordinante. Thus, Einstein's equations, i.e., $G_{ab} = 8\pi T_{ab}$, are

$$2\,\partial_z U\,\partial_z V - 3\,(\partial_z V)^2 - 2\,\partial_z\partial_z V = \tilde\rho\, e^{2U}, \tag{8}$$

$$(\partial_z V)^2 + \partial_z\partial_z V + \partial_z\partial_z U = \tilde p\, e^{2U}, \tag{9}$$

$$\partial_z V\,(\ 2\,\partial_z U + \partial_z V\ ) = \tilde p\, e^{2U}\,, \tag{10}$$

‡ Collins in Ref. [5] uses $p = (\gamma - 1)\rho$, so $\eta = (\gamma - 1)^{-1}$.

where $\tilde{\rho} = 8\pi\rho$ and $\tilde{p} = 8\pi p$.

On the other hand, $\nabla_a T^{ab} = 0$ yields

$$\partial_z \tilde{p} = -(\tilde{p} + \tilde{\rho})\,\partial_z U. \tag{11}$$

Of course, due to Bianchi's identities, equations (8), (9), (10) and (11) are not independent, so we will use only (8), (10), and (11).

By using the equation of state

$$\rho(z) = \eta\; p(z)\,, \tag{12}$$

we can write (11) as

$$\partial_z \left(\ln \tilde{p} + (\eta + 1)U\right) = 0 \tag{13}$$

and so

$$\tilde{p} = C\, e^{-(\eta+1)U}, \tag{14}$$

where C is an arbitrary constant. Thus, (8) and (10) become

$$2\; \partial_z U\, \partial_z V - 3\; (\partial_z V)^2 - 2\, \partial_z \partial_z V = \eta\, C\, e^{(1-\eta)U}, \tag{15}$$

$$\partial_z V\; (\; 2\; \partial_z U + \partial_z V\;) = C\, e^{(1-\eta)U}. \tag{16}$$

The change of variables

$$\xi = \int_{z_0}^{z} e^{(1-\eta)U(u)/2} du, \tag{17}$$

brings (15) and (16) to

$$(1+\eta)\; \partial_\xi U\, \partial_\xi V - 3\; (\partial_\xi V)^2 - 2\, \partial_\xi \partial_\xi V = \eta\, C, \tag{18}$$

$$2\; \partial_\xi U\, \partial_\xi V + (\partial_\xi V)^2 = C. \tag{19}$$

Hence

$$\partial_\xi \partial_\xi V + \frac{(\eta+7)}{4}\; (\partial_\xi V)^2 + \frac{(\eta-1)}{4}\; C = 0. \tag{20}$$

By writing

$$W(\xi) = e^{\frac{\eta+7}{4} V(\xi)}, \tag{21}$$

we can write Eq. (20) as

$$\partial_\xi \partial_\xi W = -\frac{C(\eta+7)(\eta-1)}{16}\, W\,. \tag{22}$$

Setting $\alpha = C(\eta+7)(\eta-1)/16$, we can write the general solution of (22) as

$$W(\xi) = \begin{cases} C_1\, \dfrac{\sin\sqrt{\alpha}(\xi + C_2)}{\sqrt{\alpha}} & \text{if} \quad \alpha > 0 \\ C_1\, \xi + C_1\, C_2 & \text{if} \quad \alpha = 0 \\ C_1\, \dfrac{\sinh\sqrt{-\alpha}(\xi + C_2)}{\sqrt{-\alpha}} & \text{if} \quad \alpha < 0\;, \end{cases} \tag{23}$$

where C_1 and C_2 are arbitrary constants. Notice that the three cases in (23) are covered (as a limit when $\alpha = 0$) by the first one. Therefore, taking into account (21), we can write for arbitrary α

$$V(\xi) = \ln\left(\frac{C_1\, \sin\sqrt{\alpha}(\xi + C_2)}{\sqrt{\alpha}}\right)^{\frac{4}{\eta+7}}. \tag{24}$$

Now, from (19) we can write $\partial_\xi U$ in terms of $V(\xi)$ as

$$\partial_\xi U \;=\frac{C}{2\,\partial_\xi V}-\frac{\partial_\xi V}{2}. \tag{25}$$

By integrating it, we find the general solution for $U(\xi)$

$$U(\xi) = \ln\left(\left(\frac{C_3\,\sin\sqrt{\alpha}(\xi+C_2)}{\sqrt{\alpha}}\right)^{-\frac{2}{\eta+7}}\left(\cos\sqrt{\alpha}(\xi+C_2)\right)^{-\frac{2}{\eta-1}}\right), \tag{26}$$

where C_3 is a new arbitrary constant.

If we now make the transformation: $(t,x,y,z)\to(t,x,y,\xi)$, the line element (3) becomes

$$\begin{aligned} ds^2 = &-\left(\frac{C_3\,\sin\sqrt{\alpha}(\xi+C_2)}{\sqrt{\alpha}}\right)^{-\frac{4}{\eta+7}}\left(\cos\sqrt{\alpha}(\xi+C_2)\right)^{-\frac{4}{\eta-1}}dt^2 \\ &+\left(\frac{C_1\,\sin\sqrt{\alpha}(\xi+C_2)}{\sqrt{\alpha}}\right)^{\frac{8}{\eta+7}}\left(dx^2+dy^2\right) \\ &+\left(\frac{C_3\,\sin\sqrt{\alpha}(\xi+C_2)}{\sqrt{\alpha}}\right)^{-2\frac{\eta+1}{\eta+7}}\left(\cos\sqrt{\alpha}(\xi+C_2)\right)^{-2\frac{\eta+1}{\eta-1}}d\xi^2, \end{aligned} \tag{27}$$

since from (17) we see that

$$g_{\xi\xi} = \left(\partial_\xi z\right)^2 g_{zz} = e^{(\eta-1)U}e^{2U} = e^{(\eta+1)U}. \tag{28}$$

From (14) we can write the pressure as

$$p(\xi) = \frac{C}{8\pi}\left(\frac{C_3\,\sin\sqrt{\alpha}(\xi+C_2)}{\sqrt{\alpha}}\right)^{2\frac{\eta+1}{\eta+7}}\left(\cos\sqrt{\alpha}(\xi+C_2)\right)^{2\frac{\eta+1}{\eta-1}} = \frac{C}{8\pi}\,g^{\xi\xi}\,. \tag{29}$$

Now, we define a new variable u such that

$$u = \frac{C_3^2\,\sin^2\sqrt{\alpha}(\xi+C_2)}{\alpha}, \tag{30}$$

and, without losing generality for $\eta\neq-7$, we set $C_1=C_3=\frac{\eta+7}{6}\,\kappa$, where κ is a new arbitrary constant. In terms of u, the metric (27) becomes

$$\begin{aligned} ds^2 = &-u^{-\frac{2}{\eta+7}}\,(1-\beta u)^{-\frac{2}{\eta-1}}\;dt^2+u^{\frac{4}{\eta+7}}\left(dx^2+dy^2\right) \\ &+\frac{9}{(\eta+7)^2\kappa^2}\,u^{-2\frac{\eta+4}{\eta+7}}\,(1-\beta u)^{-\frac{2\eta}{\eta-1}}\;du^2, \end{aligned} \tag{31}$$

where

$$\beta = \frac{9\,C\,(\eta-1)}{4(\eta+7)\,\kappa^2}. \tag{32}$$

The range of the coordinate u is $0\le u\le 1/\beta$ for $\beta>0$, and $0\le u<\infty$ for $\beta<0$. The transformation $f=u^{\frac{2}{\eta+7}}$ brings it to the form given by Collins in Ref. [5] §.

This family of solutions depending on three parameters C, η, and κ corresponds to a spacetime full with a static plane symmetric perfect fluid satisfying the equation of state $\rho=\eta\,p$.

§ Notice that there are some typos in the expression for g in page 2274 of Ref.[5], the correct one is $g=g_0[2BM_0af^{a-3}/(1+Bbf^a)^{1-1/b}]^{-(\gamma-1)/\gamma}$.

Notice that as $C \to 0$, i.e., the vacuum limit, $\beta \to 0$ and then (31) becomes

$$ds^2 = -u^{-\frac{2}{\eta+7}}\, dt^2 + u^{\frac{4}{\eta+7}} \left(dx^2 + dy^2\right) + \frac{9}{(\eta+7)^2\kappa^2}\, u^{-2\frac{\eta+4}{\eta+7}}\, du^2. \quad (33)$$

Setting

$$1 - \kappa\hat{z} = u^{\frac{3}{\eta+7}}, \quad (34)$$

we get

$$ds^2 = -\left(1-\kappa\hat{z}\right)^{-\frac{2}{3}}\, dt^2 + \left(1-\kappa\hat{z}\right)^{\frac{4}{3}} \left(dx^2 + dy^2\right) + d\hat{z}^2, \quad (35)$$

which is Taubs's vacuum plane solution [2] in the coordinates used in [1].

Since we are here interested in ordinary matter (i.e., $\rho > p > 0$ and so $C > 0$ and $\eta > 1$), we will analyze the solution for $\eta > 1$. However, (31) has a much wider range of validity. In fact, it is valid for every value of C and every $\eta \neq -7$. For *abnormal* matter some interesting solutions arise. However, the complete analysis turns out to be somehow involved. So, for the sake of clarity, we leave the complete study to a forthcoming publication [8], and restrict ourselves here to the case $\eta > 1$ and $C > 0$. In this case, (32) shows that $\beta > 0$.

2.1. The physical vertical distance coordinate $\hat{z}$

The physical distance from the "plane" where $u = 0$ to that sitting at u is

$$\begin{aligned}
\int_0^u \sqrt{g_{uu}}\, du' &= \frac{3}{(\eta+7)\kappa} \int_0^u u'^{\frac{3}{\eta+7}-1} \left(1-\beta u'\right)^{-\frac{\eta}{\eta-1}}\, du' \\
&= \frac{1}{\kappa}\, u^{\frac{3}{\eta+7}}\, {}_2F_1\!\left(\frac{3}{\eta+7}, \frac{\eta}{\eta-1}; 1+\frac{3}{\eta+7}; \beta u\right) \\
&= \frac{1}{\kappa}\, u^{\frac{3}{\eta+7}} \left(1-\beta\, u\right)^{-\frac{1}{\eta-1}}\, {}_2F_1\!\left(1, 1+\frac{3}{\eta+7}-\frac{\eta}{\eta-1}; 1+\frac{3}{\eta+7}; \beta u\right),
\end{aligned} \quad (36)$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function (see for example [9, 10]). The hypergeometric function in the last line is finite for $0 \leq u \leq 1/\beta$, since $c - a - b = \frac{1}{\eta-1} > 0$; furthermore [10]

$${}_2F_1\!\left(1, 1+\frac{3}{\eta+7}-\frac{\eta}{\eta-1}; 1+\frac{3}{\eta+7}; 1\right) = \frac{\Gamma(1+\frac{3}{\eta+7})\,\Gamma(\frac{1}{\eta-1})}{\Gamma(\frac{3}{\eta+7})\,\Gamma(\frac{\eta}{\eta-1})} = \frac{3(\eta-1)}{\eta+7}. \quad (37)$$

Hence, we see that as u goes from 0 to $1/\beta$, the physical distance goes from 0 to ∞.

Thus, we can write

$$1 - \kappa\hat{z} = u^{\frac{3}{\eta+7}}\, {}_2F_1\!\left(\frac{3}{\eta+7}, \frac{\eta}{\eta-1}; 1+\frac{3}{\eta+7}; \beta u\right), \quad (38)$$

and, in terms of the coordinate $\hat{z}$, (31) becomes

$$\begin{aligned}
&ds^2 = -u^{-\frac{2}{\eta+7}} \left(1-\beta u\right)^{-\frac{2}{\eta-1}}\, dt^2 + u^{\frac{4}{\eta+7}} \left(dx^2 + dy^2\right) + d\hat{z}^2, \\
&-\infty < t < \infty, \quad -\infty < x < \infty, \quad -\infty < y < \infty, \quad -\infty < \hat{z} < 1/\kappa\,.
\end{aligned} \quad (39)$$

where u $(0 \leq u \leq 1/\beta)$ is implicitly given in terms of $\hat{z}$ through (38).

For the sake of clarity, we present here an explicit example where the above expressions can be written in terms of algebraic functions. If $\eta = 5$, we can write (38) as

$$1 - \kappa\hat{z} = u^{\frac{1}{4}} \left(1-\beta\, u\right)^{-\frac{1}{4}}\, {}_2F_1\!\left(1, 0; \frac{5}{4}; \beta u\right) = \left(\frac{u}{1-\beta u}\right)^{\frac{1}{4}}, \quad (40)$$

and so

$$u = \frac{(1-\kappa\hat{z})^4}{1+\beta(1-\kappa\hat{z})^4}\,. \tag{41}$$

Therefore, for $\eta = 5$, (39) becomes

$$ds^2 = -\frac{\left(1+\beta(1-\kappa\hat{z})^4\right)^{\frac{2}{3}}}{(1-\kappa\hat{z})^{\frac{2}{3}}}\,dt^2 + \frac{(1-\kappa\hat{z})^{\frac{4}{3}}}{\left(1+\beta(1-\kappa\hat{z})^4\right)^{\frac{1}{3}}}\left(dx^2+dy^2\right) + d\hat{z}^2,$$
$$-\infty < t < \infty, \quad -\infty < x < \infty, \quad -\infty < y < \infty, \quad -\infty < \hat{z} < 1/\kappa\,. \tag{42}$$

2.2. The singularity

We see that the metric has a spacetime curvature singularity at $\hat{z} = 1/\kappa$ ($u = 0$), as it occurs for Taub's empty plane symmetric spacetime. This singularity, as it happens with the Schwarzschild's one at $r = 0$, is a *true* one in the sense that spacetime curvature diverges. For straightforward computation of the scalar quadratic in the Riemann tensor yields

$$R_{abcd}R^{abcd} = \frac{64}{27}\,\kappa^4\,u^{-\frac{12}{\eta+7}}\,(1-\beta u)^{2\frac{\eta+1}{\eta-1}}\left(1 + \frac{2(\eta+1)(\eta+3)}{3(\eta-1)}\,\beta u \right.$$
$$\left. + \frac{159+172\,\eta+142\,\eta^2+36\,\eta^3+3\,\eta^4}{12(\eta-1)^2}\,\beta^2 u^2\right)\,, \tag{43}$$

so $R_{abcd}R^{abcd} \to \infty$ as $u \to 0$ ($\hat{z} \to 1/\kappa$).

On the other hand, $R_{abcd}R^{abcd} \to 0$ as $u \to 1/\beta$ ($\hat{z} \to -\infty$) since $\frac{2\,(\eta+1)}{\eta-1} > 0$, suggesting it is asymptotically flat at spatial infinite in the $-\hat{z}$ direction.

Note that each slice of spacetime $t = t_0$ and $u = u_0$ is a Euclidean plane with metrics

$$d\ell^2 = u^{\frac{4}{\eta+7}}\left(dx^2+dy^2\right), \tag{44}$$

so its "size" contracts when $\hat{z}$ increases and becomes a point at the singularity $\hat{z} = 1/\kappa$ ($u = 0$).

Notice that, beyond the singularity (i.e., $\hat{z} \to 1/\kappa$), a mirror copy of the spacetime emerges, as it occurs at the vertex of a cone. But, as we shall show below, no geodesic can go from one to the other.

2.3. The pressure

From (29), the pressure can be written as

$$p(u) = \frac{C}{8\pi}\,u^{\frac{\eta+1}{\eta+7}}\,(1-\beta u)^{\frac{\eta+1}{\eta-1}}\,, \tag{45}$$

so $p(0) = p(1/\beta) = 0$. It follows readily from (38) that it vanishes like $(1-\kappa\hat{z})^{\frac{\eta+1}{3}}$ as $\hat{z} \to 1/\kappa$ ($u = 0$), and like $(1-\kappa\hat{z})^{-(\eta+1)}$ as $\hat{z} \to -\infty$ ($u \to 1/\beta$). On the other hand, it has a maximum at

$$u = \frac{(\eta-1)}{2\,\beta\,(\eta+3)} = \frac{2\,\kappa^2\,(\eta+7)}{9\,C\,(\eta+3)}\,. \tag{46}$$

In Fig. 1, we show a plot of $p(\hat{z})$.

Since $p = 0$ (and $\rho = 0$) at the singularity, so $T_{ab} = 0$ there, we see that it is an *empty* (without matter) singularity. Thus, the attraction of the infinite amount of ordinary matter curves spacetime in such a way that a singularity arises in a place free of matter.

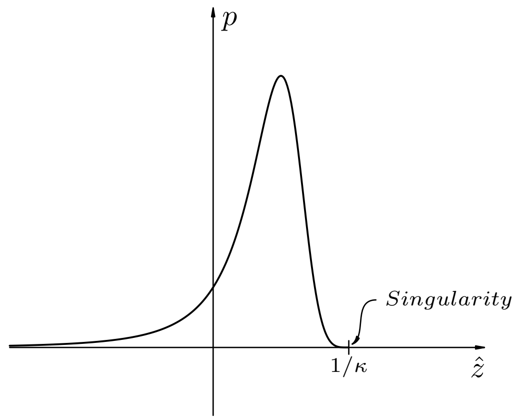

Figure 1. The pressure.

3. The geodesics

We want to study the geodesics in this spacetime. Since the metric is independent of t, x and y, the momentum covector components p_t, p_x and p_y are constant along the geodesics. For timelike geodesics, we choose τ to be the proper time; and for null ones, we choose τ to be an affine parameter. So, we can write

$$\left(\frac{du}{d\tau}\right)^2 = \frac{(\eta+7)^2\kappa^2}{9}\, u^{-\frac{6}{\eta+7}+2}\,(1-\beta u)^{\frac{2\eta}{\eta-1}} \times \left[u^{\frac{2}{\eta+7}}\,(1-\beta u)^{\frac{2}{\eta-1}}\,\tilde{E}^2 - \mu - u^{-\frac{4}{\eta+7}}\,(\tilde{p_x}^2+\tilde{p_y}^2)\right], \tag{47}$$

$$\frac{dt}{d\tau} = u^{\frac{2}{\eta+7}}\,(1-\beta u)^{\frac{2}{\eta-1}}\,\tilde{E}, \tag{48}$$

$$\frac{dx}{d\tau} = u^{-\frac{4}{\eta+7}}\,\tilde{p}_x, \tag{49}$$

$$\frac{dy}{d\tau} = u^{-\frac{4}{\eta+7}}\,\tilde{p}_y, \tag{50}$$

where $\mu = 1$, $\tilde{E} = -p_t/m$, $\tilde{p_x} = p_x/m$ and $\tilde{p_y} = p_y/m$ for timelike geodesics; and $\mu = 0$, $\tilde{E} = -p_t$, $\tilde{p_x} = p_x$ and $\tilde{p_y} = p_y$ for null ones.

The right hand side of (47) cannot be negative. Thus,

$$u^{\frac{6}{\eta+7}}\,(1-\beta u)^{\frac{2}{\eta-1}}\,\tilde{E}^2 - u^{\frac{4}{\eta+7}}\,\mu - (\tilde{p_x}^2+\tilde{p_y}^2) > 0\,. \tag{51}$$

Therefore, only vertical null geodesics touch the singularity at $u = 0$ ($\hat{z} \to 1/\kappa$) and bounce following its travel to $u = 1/\beta$ ($\hat{z} = -\infty$). Whereas, as it is shown below, non-vertical null ones as well as massive particles bounce before getting to it and bounce again before reaching $\hat{z} = -\infty$.

For the former case, the geodesic equation can be integrated in closed form. Indeed, when $\mu = 0$ and $\tilde{p_x}^2 + \tilde{p_y}^2 = 0$, we get from (47) and (48)

$$\frac{dt}{du} = \pm\frac{3}{(\eta+7)\kappa}\,u^{\frac{4}{\eta+7}-1}\,(1-\beta u)^{-1}\,. \tag{52}$$

So

$$|t-t_0| = \frac{3}{4\kappa}\,u^{\frac{4}{\eta+7}}\,{}_2F_1\left(\frac{4}{\eta+7}, 1; 1+\frac{4}{\eta+7}; \beta u\right). \tag{53}$$

Now, taking into account that when $z \to 1$,

$$ {}_2F_1(a,b;a+b;z) \to -\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \ln(1-z) \ , \qquad (54) $$

we see that

$$ |t-t_0| \sim \begin{cases} \dfrac{3}{4\kappa}\, u^{\frac{4}{\eta+7}} \to 0 & \text{as} \quad u \to 0 \\ -\dfrac{3\ \beta^{-\frac{4}{\eta+7}}}{(\eta+7)\kappa}\ \ln(1-\beta u) \to \infty & \text{as} \quad u \to 1/\beta\,. \end{cases} \qquad (55) $$

Therefore, we can fill the whole spacetime with never-stopping future-oriented vertical null geodesics, all of them starting and finishing at $\hat{z} = -\infty$.

It follows from (51) that the movement of non-vertical photons or massive particles is constrained to the region where $\tilde{E}^2 > \mathcal{V}(u)$, being

$$ \mathcal{V}(u) = \frac{u^{\frac{4}{\eta+7}}\, \mu \, + \, (\tilde{p_x}^2 + \tilde{p_y}^2)}{u^{\frac{6}{\eta+7}}\, (1-\beta u)^{\frac{2}{\eta-1}}} \ . \qquad (56) $$

Clearly $\mathcal{V}(u)$ is a positive continuous function of u for $0 < u < 1/\beta$ and, since $\eta > 1$, $V(u) \to +\infty$ when $u \to 0$ or $u \to 1/\beta$. Moreover we can write

$$ \mathcal{V}''(u) = 2\ \frac{\Big(u^{\frac{4}{\eta+7}}\, \mu\, P_1(\beta u) + \, (\tilde{p_x}^2 + \tilde{p_y}^2)\, P_2(\beta u)\Big)}{u^{\frac{6}{\eta+7}+2}(1-\beta u)^{\frac{2}{\eta-1}+2}\ (\eta-1)^2\, (\eta+7)^2} \ , \qquad (57) $$

where $P_1(z)$ and $P_2(z)$ are the second degree polynomials in z:

$$ P_1(z) = 2\, z\Big(5 + \eta\,(\eta+10)\Big)\Big(z(\eta+3) - (\eta-1)\Big) + \, (\eta-1)^2\, (\eta+9) \ , \qquad (58) $$

$$ P_2(z) = 2\, z\Big(1 + \eta\,(\eta+14)\Big)\Big(2z(\eta+1) - 3(\eta-1)\Big) + \, 3\ (\eta-1)^2\, (\eta+13) \ . \qquad (59) $$

We can readily see that both polynomials are positive definite in $0 < z < 1$ for $\eta > 1$. Then $\mathcal{V}''(u) > 0$, and so $\mathcal{V}(u)$ has one and only one minimum $\mathcal{V}_0$. Therefore for any value of $\tilde{E}^2$ greater than $\mathcal{V}_0$ we find two (and only two) turning points where $\tilde{E}^2 = \mathcal{V}(u)$. Thus timelike geodesics as well as non-vertical null ones oscillate between two planes determined by initial conditions.

As no geodesic can cross the singularity we can think of it by removing the point $\hat{z} = 1/\kappa$ as the boundary of the spacetime.

Hence, the attraction of the *infinite* amount of matter filling the spacetime with plane symmetry shrinks the spacetime in such a way that it finishes at the empty singular boundary.

4. Concluding remarks

We have done a detailed study of the Collins exact solution of Einstein's equations corresponding to a static and plane symmetric distribution of ordinary matter with density proportional to the pressure.

This simple spacetime turns out to present some somehow astonishing properties. Namely, the attraction of the *infinite* amount of matter filling the spacetime with plane symmetry shrinks the spacetime in such a way that it finishes at an empty singular boundary.

Only vertical null geodesics touch the boundary and bounce following its travel to $-\infty$, whereas non-vertical null ones as well as massive particles bounce before getting to it and oscillate between two planes.

Since the density vanishes at the singularity this example shows that, as suggested in [1], repelling singularities can arise in a place free of matter.

For *abnormal* matter (*i.e.* $\eta < 1$ or $C < 0$) some interesting solutions arise, depending on the combination of the parameters C and η. For instance, for $C < 0$ and $\eta > 1$ the spacetime is bounded between two singularities of this kind separated by a finite distance [8].

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